# Centers of KLR algebras and cohomology rings of quiver varieties

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*Abstract.* Attached to a weight space in an integrable highest weight representation of a simply-laced Kac-Moody algebra $\mathfrak{g}$, there are two natural commutative algebras: the cohomology ring of a quiver variety and the center of a cyclotomic KLR algebra. In this note, we describe a natural geometric map between these algebras in terms of quantum coherent sheaves on quiver varieties.

The cohomology ring of an algebraic symplectic variety can be interpreted as the Hochschild cohomology of a quantization of this variety in the sense of Bezrukavnikov and Kaledin. On the other hand, cyclotomic KLR algebras appear as Ext-algebras of certain particular sheaves, and thus its center receives a canonical map from the Hochschild cohomology of the category. We show that this map is an isomorphism in finite type, and injective in general. We further note that the Kirwan surjectivity theorem for quivers of finite type is an easy corollary of these results.

The most important property of this map is its compatibility with actions of the current algebra on both the cohomology of quiver varieties and on the Hochschild cohomology of any category with a categorical action of $\mathfrak{g}$. The structure of these current algebra actions allow us to show the desired results.

Let $\mathfrak{g}$ be a simply-laced Kac-Moody algebra. For each pair of weights $\lambda, \mu$ such that $\mu \leq \lambda$, we have a quiver variety $\mathfrak{M}^\lambda_\mu$. Nakajima [Nak98] has shown that the middle degree cohomology of $\mathfrak{M}^\lambda_\mu$ is isomorphic to the $\mu$-weight of the representation with highest weight $\lambda$, making this variety a geometric avatar of the weight space. On the other hand, we also have a cyclotomic KLR algebra $R^\lambda_\mu$, which provides a categorification of this weight space, in the sense that $K^0(R^\lambda_\mu)$ is isomorphic to an integral form of this weight space, with the classes of indecomposable projectives matching the canonical basis.

Attached to these objects, we have a pair of commutative algebras: the cohomology ring $H^*(\mathfrak{M}^\lambda_\mu)$ of the quiver variety, and the center $Z(R^\lambda_\mu)$ of the KLR algebra. It is clear from various analogies that these algebras should be closely related. For example, both have actions of a current algebra, and both are isomorphic to dual Weyl modules for this current algebra in finite type. Also, in type A, explicit calculations show that they are isomorphic as algebras.

Our goal is to show that these algebras are isomorphic for any ADE type quiver variety and to give a natural, geometric isomorphism between them. For other simply-laced types, we show that there is an injection $H^*(\mathfrak{M}^\lambda_\mu) \hookrightarrow Z(R^\lambda_\mu)$.

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Constructing this map requires the theory of quantizations of quiver varieties. We'll follow the notation of [Weba] throughout. In that work, we concentrated our interest in a quantization $\mathcal{A}_\mu$ of the structure sheaf of $\mathfrak{M}^\lambda_\mu$ compatible with its symplectic structure. These quantizations are indexed by **periods**, and we'll specialize throughout to the case where the period is integral.

There is a categorical action of the Lie algebra $\mathfrak{g}$ on the coherent modules over the algebra $\mathcal{A}_\mu$ for all the different weights $\mu$ appearing in the weight decomposition of the representation with highest weight $\lambda$ [Weba]. Inside the derived category of $\mathcal{A}_\mu$-modules, there's a natural subcategory $C$, which is the smallest invariant subcategory containing the constant sheaf on $\mathfrak{M}^\lambda_\lambda$. This can also be geometrically characterized as the subcategory of objects with compact support. In [Webc, Th. A], we showed an equivalence of dg-categories $C \cong R^\lambda_\mu$-dg-mod, where $R^\lambda_\mu$ is the cyclotomic KLR algebra attached to the same Lie algebra and pair of weights.

We have a natural pullback map on the Hochschild cohomology of $\mathcal{A}_\mu$-modules to that of $C$. Put in more down-to-earth terms, this gives a map $\phi\colon H^*(\mathfrak{M}^\lambda_\mu) \to Z(R^\lambda_\mu)$, realizing the former as the Hochschild homology of the category of $\mathcal{A}$-modules, as in [BLPW, §5.4], and the latter as the Ext-algebra of a semi-simple object in $C$.

The main result of this note is:

**Theorem A** (Theorem 3.4)**.** *If $\mathfrak{g}$ is type ADE, the map $\phi$ is an algebra isomorphism. If $\mathfrak{g}$ is of general simply-laced type, then $\phi$ is injective.*

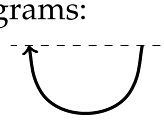
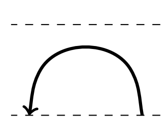
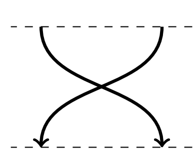
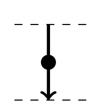
The key fact which allows us to prove this is that $\phi$ is compatible with an action of the current algebra $\mathfrak{g}[t]$ on the cohomology rings $H^*(\mathfrak{M}^\lambda_\mu)$ was defined by Varagnolo [Var00] and on the center $Z(R^\lambda_\mu)$ by the author, Beliakova, Habiro and Lauda [BHLWa] and Shan, Vasserot and Varagnolo [SVV], independently. Once we know that $\phi$ commutes with the current algebra action, the isomorphism of both source and target with a dual Weyl module shows that $\phi$ is an isomorphism.

## 1. Categorifications and current algebras

First, we will need some general background on categorical actions.

1.1. **The 2-category $\mathcal{U}$.** Fix an oriented graph $\Gamma$, which we will assume is simply-laced. The object of interest for this subsection is a strict 2-category $\mathcal{U}$, due to Khovanov and Lauda [KL10]. We will give a more compact definition of this category, shown to be equivalent to that of earlier literature such as [KL10, CL15, Webb] in a recent paper of Brundan [Bru].

In order to define it, we will need to define a class of diagrams. Consider the set of diagrams in the horizontal strip $\mathbb{R} \times [0, 1]$ composed of embedded oriented curves, whose endpoints lie on distinct points of $\mathbb{R}\times\{0\}$ and $\mathbb{R}\times\{1\}$. At each point, projection to the $y$-axis must locally be a diffeomorphism, unless at that point it looks like one of the diagrams:

$\iota =$  $\epsilon =$  $\psi =$  $y =$ 

We'll consider labelings of the components of these diagrams by elements of $\Gamma$. The **top** of such a diagram is the sequence where we read off the label of each of the endpoints on $\mathbb{R} \times \{1\}$ in order from left to right, taking them with positive sign if the curve is oriented upward there, and a negative sign if it is oriented downward. The **bottom** is defined similarly with the endpoints on $\mathbb{R} \times \{0\}$. The **vertical composition** $ab$ of two diagrams where the bottom of $a$ matches the top of $b$ is the stacking of $a$ on top of $b$ and then scaling the $y$-coordinate by $1/2$ to lie again in $\mathbb{R} \times [0, 1]$. The horizontal composition of two diagrams $a \circ b$ places $a$ to the *right* of $b$ in the plane, and thus has the effect of concatenating their tops and bottoms in the opposite of the usual order.

We let $\lambda^i = \alpha_i^\vee(\lambda)$ for any weight $\lambda$.

**Definition 1.1.** *Let $\tilde{\tilde{\mathcal{U}}}$ be the strict 2-category where*

- *the set of objects is the weight lattice of the Kac-Moody algebra $\mathfrak{g}_\Gamma$.*
- *1-morphisms $\mu \to \nu$ are sequences $\mathbf{i} = (i_1, \ldots, i_m)$ with each $i_j \in \pm\Gamma$, which we interpret as a list of simple roots and their negatives such that $\mu + \sum_{j=1}^m \alpha_{i_j} = \nu$. Composition is given by concatenation.*
- *2-morphisms $h \to h'$ between sequences are* **k***-linear combinations of diagrams of the type defined above with $h$ as bottom and $h'$ as top.*

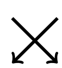
Since the underlying objects in $\tilde{\tilde{\mathcal{U}}}$ are fixed for any 2-morphism, we incorporate them into the diagram by labeling each region of the place with $\mu$ at the far left, $\nu$ at the far right, and intermediate regions are labeled by the rule

$\mu$  $\mu - \alpha_i$ .

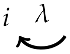
$i$

We'll typically use $\mathcal{E}_i$ to denote the 1-morphism $(i)$ (leaving the labeling of regions implicit) and $\mathcal{F}_i$ to denote $(-i)$.

We can define a **degree** function on diagrams. The degrees are given on elementary diagrams by

deg  $= \begin{cases} -2 & i = j \\ 1 & i \leftrightarrow j \\ 0 & i \not\leftrightarrow j \end{cases}$ deg  $= 2$

$i \quad j$ $\qquad\qquad\qquad\qquad i$

deg $^{i\ \lambda}$  $= \langle \lambda, \alpha_i \rangle - 1$ deg  $_{i\ \lambda} = -\langle \lambda, \alpha_i \rangle - 1.$

For a general diagram, we sum together the degrees of the elementary diagrams it is constructed from. This defines a grading on the 2-morphism spaces of $\tilde{\tilde{\mathcal{U}}}$.

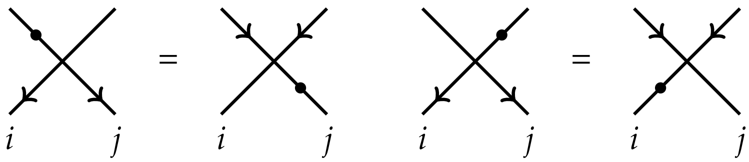
Consider the polynomials

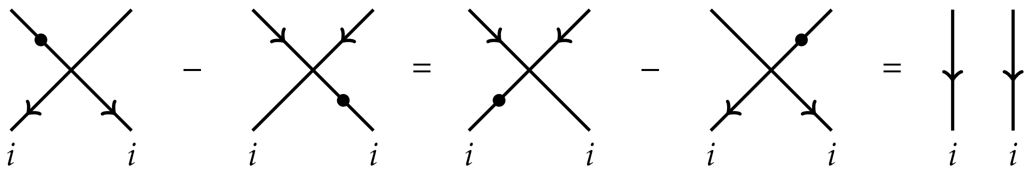
$$Q_{ij}(u, v) = (u - v)^{\#\{j \to i\}}(v - u)^{\#\{i \to j\}}$$

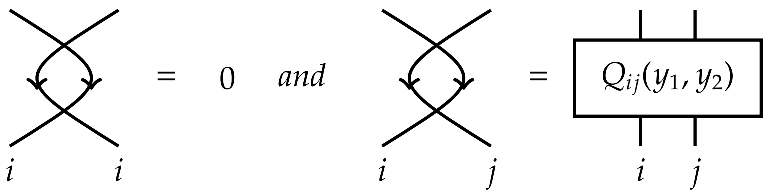
**Definition 1.2.** *Let $\mathcal{U}$ be the quotient of $\tilde{\tilde{\mathcal{U}}}$ by the following relations on 2-morphisms:*

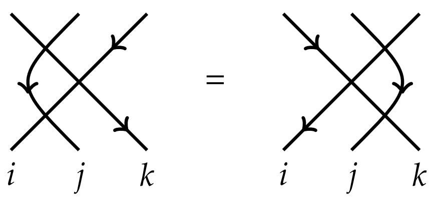
- *$\epsilon$ and $\iota$ are the units and counits of an adjunction, i.e. critical points can cancel.*

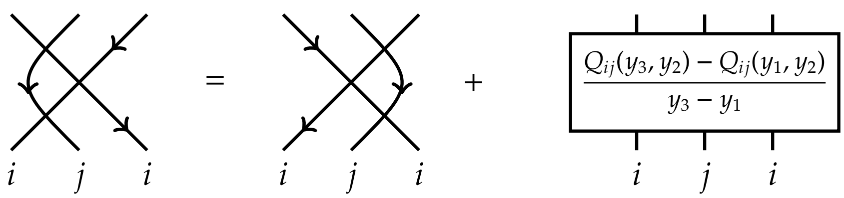
- *the endomorphisms of words only using $\mathcal{F}_i$ (or by duality only $\mathcal{E}_i$'s) satisfy the relations of the* **quiver Hecke algebra** $R$.

(1a)  *unless $i = j$*

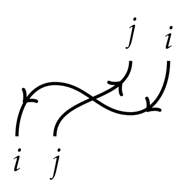
(1b) 

(1c) 


(1d)  *unless $i = k \neq j$*

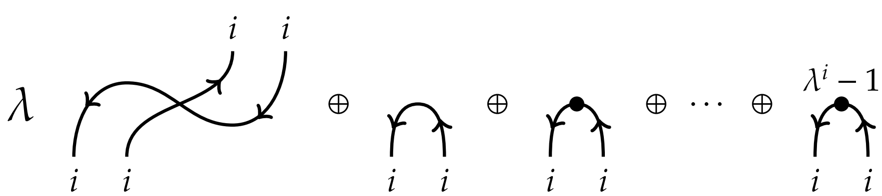
(1e) 


- *the composition*



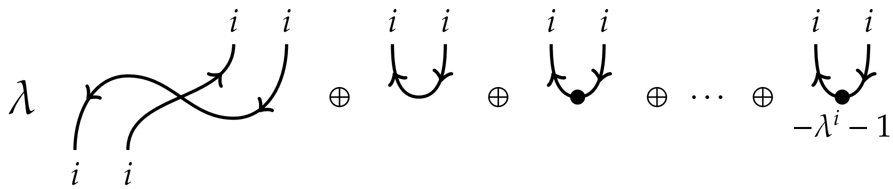
*possesses an inverse.*

- *if $\lambda^i \geq 0$, then the map $\sigma_{\lambda,i} \colon \mathcal{E}_i\mathcal{F}_i \to \mathcal{F}_i\mathcal{E}_i \oplus \mathrm{id}_\lambda^{\oplus \lambda^i}$ given by*



*possesses an inverse.*

- *if* $\lambda^i \leq 0$, *then the map* $\sigma_{\lambda,i}\colon \mathcal{E}_i\mathcal{F}_i \oplus \mathrm{id}_\lambda^{\oplus -\lambda^i} \to \mathcal{F}_i\mathcal{E}_i$ *given by*




*possesses an inverse.*

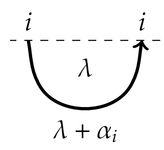
1.2. **Dualities.** In this category, the functors $\mathcal{E}_i$ and $\mathcal{F}_i$ are biadjoint up to shift. Some care about this biadjunction is needed, since it is not unique up to isomorphism. However, since the degree 0 automorphisms of $\mathcal{F}_i$ are simply a copy of the scalars, the biadjunction is unique up to scalar multiplication.

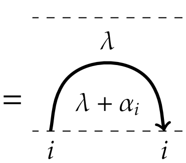
Choosing such a biadjunction between $\mathcal{E}_i 1_\lambda$ and $\mathcal{F}_i 1_{\lambda+\alpha_i}$ for each $i$ and $\lambda$ defines a duality functor on $\boldsymbol{\mathcal{U}}$ such that

$$(\mathcal{E}_i 1_\lambda)^\star = \mathcal{F}_i 1_{\lambda+\alpha_i}(\langle \lambda, \alpha_i\rangle + 1) \qquad (\mathcal{F}_i 1_{\lambda+\alpha_i})^\star = \mathcal{E}_i 1_\lambda(-\langle \lambda, \alpha_i\rangle - 1)$$

and $u^\star$ is right adjoint to $u$. In this case, $u = u^{\star\star}$ up to shift for every 1-morphism $u$.

We can define one such duality by defining

$\iota' =$ 

$\epsilon' =$ 


according to the rule of [Bru, (1.14-18)]. We call this the **Cautis-Lauda duality**, since it is uniquely characterized by the relations in [CL15, §2]. However, this choice of duality is not cyclic; that is, the double dual of a morphism might not coincide with the original morphism. To define an action on $Z(R^\lambda_\mu)$, we need a cyclic duality. Such a duality is constructed in [BHLWa, §9], and will be discussed in greater detail in [BHLWb]. For our purposes, it will be useful to give a self-contained account here.

To start with, we can define other dualities by how they differ from the Cautis-Lauda duality.

**Definition 1.3.** *Given a map* $\theta\colon \mathcal{X} \to \mathbf{k}^\times$, *and a biadjunction* $(\mathcal{E}_i, \mathcal{F}_i, \epsilon, \iota, \epsilon', \iota')$, *we let the twist of this biadjunction be the biadjunction which leaves* $\epsilon, \iota$ *unchanged and takes* $\tilde{\epsilon}'_i\colon \mathcal{F}_i\mathcal{E}_i 1_\lambda \to 1_\lambda$ *to be* $\theta(\lambda)\epsilon'$; *consequently, we must take* $\tilde{\iota}'_i\colon 1_\lambda \to \mathcal{E}_i\mathcal{F}_i 1_\lambda$ *to be* $\theta(\lambda+\alpha_i)^{-1}\iota'_i$.

It will be useful for us to sometimes use partially defined maps, since for an indecomposable categorical module, there will only be non-zero categories associated to a single coset of the root lattice $\mathcal{Y}$ in the weight lattice $\mathcal{X}$.

Now, let $\beta\colon \mathcal{Y} \times \mathcal{Y} \to \mathbb{Z}/2\mathbb{Z}$ be a bilinear map of abelian groups such that we have

$$\beta(\gamma, \delta) + \beta(\delta, \gamma) \equiv \langle \gamma, \delta\rangle \pmod 2.$$

Such $\beta$ obviously form an affine space over the abelian group of symmetric $\mathbb{Z}/2\mathbb{Z}$-valued forms on $\mathcal{Y}$. Note that we can very easily show that this space is non-empty

by picking an order on roots, and declaring that

$$\beta(\alpha_i,\alpha_j)=\begin{cases}\langle\alpha_i,\alpha_j\rangle & i>j\\ 0 & j\leq i.\end{cases}$$

In [Var00], such a function is produced by a pair of maps $(-)_\pm\colon \mathcal{Y}\to\mathcal{Y}$ such that $\lambda=\lambda_++\lambda_-$ and $\langle\lambda_+,\mu\rangle=\langle\lambda,\mu_-\rangle$, and setting $\beta(\gamma,\delta)=\langle\gamma_+,\delta\rangle \pmod 2$. In [BHLWa], we work by assuming that $\Gamma=\Gamma_0\cup\Gamma_1$ is bipartite, and take

$$\beta(\alpha_i,\alpha_j)=\begin{cases}\langle\alpha_i,\alpha_j\rangle & i\in\Gamma_0\\ 0 & i\in\Gamma_1\end{cases}$$

Pick a map $\Pi\colon\mathcal{X}\to\mathcal{Y}$ compatible with the $\mathcal{Y}$ action by addition. Let the $\beta$-Cautis-Lauda duality be that obtained by twisting the Cautis-Lauda duality by the function $\mu\mapsto(-1)^{\beta(\Pi(\mu),\alpha_i)}$.

**Proposition 1.4.** *For any such $\beta$, the $\beta$-Cautis-Lauda duality is cyclic.*

*Proof.* The Cautis-Lauda duality fails to be cyclic because the crossing $\psi$ of differently colored strands may not be. In fact, in our conventions, a full counter clockwise rotation of this diagram differs from the original by a factor of $v_{ij}=t_{ij}/t_{ji}=(-1)^{\langle\alpha_i,\alpha_j\rangle}$. In [CL15, 2.5], the left-hand side of the equation changes by $-1$ raised to

$$\begin{aligned}\beta(\lambda,\alpha_j)+\beta(\lambda+\alpha_j,\alpha_i)-2\beta(\lambda,\alpha_i)-\beta(\lambda+\alpha_i,\alpha_j)&=\beta(\alpha_i,\alpha_j)-2\beta(\alpha_j,\alpha_i)\\&=\langle\alpha_i,\alpha_j\rangle.\end{aligned}$$

Thus, $\psi$ is cyclic in our geometric biadjoint. It's clear that the cyclicity of $y$ remains unchanged, so the result is proved. ☐

**Proposition 1.5** ([BHLWa, 5.1] )**.** *The elements of* $\mathrm{Tr}(\mathcal{U})$ *defined by:*

$$\mathsf{E}_{i,r}1_\lambda:=\left[\begin{array}{ccc} & i & \\ \lambda & \overset{\uparrow}{\bullet} & r\\ & i & \end{array}\right],\qquad \mathsf{F}_{j,s}1_\lambda:=\left[\begin{array}{ccc} & i & \\ \lambda & \underset{\downarrow}{\bullet} & r\\ & i & \end{array}\right],\qquad \mathsf{H}_{i,r}1_\lambda:=\left[p_{i,r}(\lambda)\colon\ \mathrm{id}_\lambda\to\mathrm{id}_\lambda\right],$$

*define a homomorphism*

$$\rho\colon \dot{\mathbf{U}}(\mathfrak{g}[t])\longrightarrow \mathrm{Tr}(\mathcal{U}), \tag{2}$$

*given by*

$$x^+_{i,r}1_\lambda\mapsto\mathsf{E}_{i,r}1_\lambda,\qquad x^-_{j,s}1_\lambda\mapsto\mathsf{F}_{j,s}1_\lambda,\qquad \xi_{i,r}1_\lambda\mapsto\mathsf{H}_{i,r}1_\lambda. \tag{3}$$

As explained in [BHLWa, §9], this map induces an action of the current algebra on the cocenter of any category $C$ with a categorical $\mathfrak{g}$-action. Furthermore, if we choose a cyclic duality, then we also have an action on the center by wrapping with bubbles, as in the graphical calculus of [CW10]. More precisely, if we have a 1-morphism $u\colon\mu\to\nu$, and an endomorphism $a\colon u\to u$, then we can define the convolution $a\star -\colon Z(C_\mu)\to Z(C_\nu)$ given by

$$a\star z=\epsilon_u\circ(a\otimes z\otimes 1_{u^\star})\circ\iota_{u^\star}$$

It will sometimes be useful to factor this convolution into the natural map $\mathrm{End}(u,u)\otimes Z(C_\mu) \to \mathrm{End}(u,u)$ and then a trace map $\tau\colon \mathrm{End}(u,u) \to Z(C_\nu)$ given by $\tau(a) = a \star 1_\mu$. In this notation, the action of the current algebra is given by $x^{\pm}_{i,r}(z) = y^r_{\pm i} \star z$ where $y_{\pm i}$ is the dot endomorphism of $\mathcal{E}_{\pm i}$. For simplicity, we let $\tau^\star(a) = \tau(a^\star)$.

1.3. **Cyclotomic quotients.** One of the actions of this 2-category that interests us is on the modules over cyclotomic KLR algebras $R^\alpha_\mu$ and their natural deformations $\check{R}^\lambda_\mu$ defined by Khovanov and Lauda [KL09] and Rouquier [Roua, Roub]. These play a universal role amongst all categorifications of the simple representations with a given highest weight. We refer to [Webb, §3.3 & 3.5] for their definition, and we will follow the notation of that paper. As mentioned above, the center and cocenter of these algebras inherit a current algebra action, analyzed extensively in [SVV]. We'll say that **algebraic Kirwan surjectivity** holds for a given quiver if the map $\kappa_a\colon Z(R_{\lambda-\mu}) \to Z(R^\lambda_\mu)$ for all $\mu$. This follows immediately if the sum $\oplus_\mu Z(R^\lambda_\mu)$ is generated as a module over $\mathfrak{g}[t]$ by the identities $1_\mu$ in $R^\lambda_\mu$, since the image of $\kappa_a$ is closed under the current algebra action.

**Proposition 1.6.** *Algebraic Kirwan surjectivity holds for finite type ADE.*

*Proof.* This is essentially equivalent to [BHLWa, 7.3] or [SVV, Th. 1]. By [KN12], the dual Weyl module has a unique simple quotient which is a copy of $W_{\lambda_{min}}$ where $\lambda_{min}$ is a dominant weight of minimal norm such that $\lambda_{min} \leq \lambda$. Furthermore, this copy has degree 0, and the quotient map kills all elements of positive degree in $Z(R^\lambda_{\lambda_{min}})$. Thus, the identity of this algebra has non-zero image in this simple quotient. This shows that it is a generator, since it is not contained in the unique maximal submodule. □

## 2. The quantum geometry of quiver varieties

The other categorical representation that will interest us is a geometric one arising in [Weba]. This representation $\mathcal{G}_\lambda$ depends on a choice of a highest weight $\lambda$ used in the definition of the underlying quiver varieties. This representation is a direct quantization/categorification of Nakajima's construction in the sense that $\mathcal{E}_i, \mathcal{F}_i$ are sent to functors of convolutions with sheaves $\mathscr{E}_i, \mathscr{F}_i$ defined in [Weba, (3.1–3.2)]. The sheaf $\mathscr{E}_i$ is a quantization of the structure sheaf on the Hecke correspondence $\mathfrak{P}_i$ used by Nakajima, which is a smooth Lagrangian subvariety of $\mathfrak{M}^\lambda_{\mu+\alpha_i} \times \mathfrak{M}^\lambda_\mu$.

Assume that $i$ is a source of $\Gamma$. The functors $\mathscr{E}_i, \mathscr{F}_i$ are constructed as Hamiltonian reductions of the convolution on D-modules along the correspondence

$$\hat{X}^\lambda_{\mu-\alpha_i} \longleftarrow \hat{X}^\lambda_{\mu;\alpha_i} \longrightarrow \hat{X}^\lambda_\mu.$$

where $\hat{X}^\lambda_\mu$ is the space of framed quiver representations on a fixed underlying vector space, modulo $GL(\mathbb{C}^{v_i})$, as defined in [Weba, 3.11]. One can view the points of this space as a quiver representation on $\Gamma \setminus \{i\}$ and choice of subspace $V_i$ of dimension $v_i$ in $V_{out} \cong \sum_{i\to j} V_j$. The space $\hat{X}^\lambda_{\mu;\alpha_i}$ is the analogous space with a nested pair $V_i \subset V'_i$ with dimensions $v_i$ and $v_i + 1$. The projections forgetting either of these spaces is proper and smooth with fibers given by projective spaces. Thus, Poincaré duality on the fibers with the complex orientation induces a biadjunction of $(\mathscr{E}_i, \mathscr{F}_i)$, and so

a duality on the image of $\mathcal{U}$ under $\mathcal{G}_\lambda$. We call this the **geometric duality**[2]. As with the Cautis-Lauda duality, the $\beta$-geometric duality is twist of the geometric duality by $\mu \mapsto (-1)^{\beta(\mu-\lambda,\alpha_i)}$.

Every other weight that appears in this categorification is of the form $\mu = \lambda - \sum v_i\alpha_i$. We'll consider the function on this coset that sends $\xi_i(\mu) = \mu^i - v_i$. Note that this is the same as the number of times $F_i$ must be applied to bring $\mu$ to the boundary of the Weyl polytope. In finite type, this is the same as $m_i$ where $\mu = w_0\lambda + \sum_j m_j\alpha_j$.

**Proposition 2.1.** *The representation $\mathcal{G}_\lambda$ sends the Cautis-Lauda duality to the geometric duality twisted by the function $\mu \mapsto (-1)^{\xi_i(\mu)}$.*

*Proof.* In order to calculate the twist, we need only calculate the value of one non-zero diagram using both dualities and compare these. Choose a weight $\mu$; assume for now that $\mu^i \leq 0$, and let $m_i = \xi(\mu) = v_i + \mu^i$. In this case, in the Cautis-Lauda duality, we have that $\epsilon_i'(1 \otimes y^{-\mu^i+1})\iota_i = 1$ on $1_{\mu-\alpha_i}$. Thus, we must compute this value in the geometric duality and see that we obtain $(-1)^{m_i}$. That is, we must compute the convolution $(q_1)_*q_2^*(c_1(V_i'/V_i)^{-\mu^i+1} \cup (q_2)_*q_1^*1)$ using the diagram

$$\hat{X}^\lambda_{\mu-\alpha_i} \xleftarrow{q_1} \hat{X}^\lambda_{\mu;\alpha_i} \xrightarrow{q_2} \hat{X}^\lambda_{\mu;\alpha_i} \times_{\hat{X}^\lambda_\mu} \hat{X}^\lambda_{\mu;\alpha_i}.$$

The normal bundle of $\hat{X}^\lambda_{\mu;\alpha_i}$ in $\hat{X}^\lambda_{\mu;\alpha_i} \times_{\hat{X}^\lambda_\mu} \hat{X}^\lambda_{\mu;\alpha_i}$ is given by tangent vectors giving variations of $V_i'$ with $V_{out} \supset V_i' \supset V_i$; this space is isomorphic to $\mathrm{Hom}(V_i'/V_i, V_{out}/V_i')$; thus, letting $z := c_1(V_i'/V_i)$, the resulting Euler class is

$$e(V_i'/V_i, V_{out}/V_i') = (-z)^{m_i-1} + (-z)^{m_i-2}c_1(V_{out}/V_i') + \cdots + c_{v_i-1}(V_{out}/V_i').$$

Thus, we have that

$$z^{-\mu^i+1} \cup (q_2)_*q_1^*1 = (-1)^{m_i}(z^{v_i} - z^{v_i-1}c_1(V_{out}/V_i') + \cdots + (-1)^{m_i+1}z^{-\mu_i}c_{v_i-1}(V_{out}/V_i')).$$

The fibers of the map $q_1$ are isomorphic to projective spaces $\mathbb{P}^{v_i}$; let $j$ be the inclusion of such a fiber. The restriction of $V_{out}/V_i'$ to a fiber is trivial, so $j^*c(V_{out}/V_i') = 1$, while $V_i'/V_i$ pulls back to $\mathcal{O}(1)$, so $j^*z = H$, the standard hyperplane class. Thus

$$j^*(z^{-\mu^i} \cup (q_2)_*q_1^*1) = (-1)^{m_i}H^{v_i}.$$

The integral of this over the fiber is, of course, $(-1)^{m_i}$, so we have that

$$(q_1)_*q_2^*(c_1(V_i'/V_i)^{-\mu^i} \cup (q_2)_*q_1^*1) = (-1)^{m_i}$$

as desired.

The calculation when $m_i \geq 0$ is similar. In this case, we use that $\epsilon_i'(1 \otimes y^{\mu^i+1})\iota_i = 1$ on $1_{\mu+\alpha_i}$. That is, we must compute the convolution $(r_1)_*r_2^*(z^{\mu^i+1} \cup (r_2)_*r_1^*1)$ using the diagram

$$\hat{X}^\lambda_{\mu+\alpha_i} \xleftarrow{r_1} \hat{X}^\lambda_{\mu+\alpha_i;\alpha_i} \xrightarrow{r_2} \hat{X}^\lambda_{\mu+\alpha_i;\alpha_i} \times_{\hat{X}^\lambda_\mu} \hat{X}^\lambda_{\mu+\alpha_i;\alpha_i}.$$

[2] Of course "Poincaré duality" or "Verdier duality" would both be very appropriate names, but could easily lead to confusion.

The normal bundle to $\hat{X}^\lambda_{\mu+\alpha_i;\alpha_i}$ in $\hat{X}^\lambda_{\mu+\alpha_i;\alpha_i} \times_{\hat{X}^\lambda_\mu} \hat{X}^\lambda_{\mu+\alpha_i;\alpha_i}$ corresponds to variations of $V_i \subset V_i'$ and thus is given by $\mathrm{Hom}(V_i, V_i'/V_i)$ so we have that

$$z^{\mu^i+1} \cup (r_2)_* r_1^* 1 = z^{m_i} - z^{m_i-1} c_1(V_{out}/V_i') + \cdots + (-1)^{m_i-1} z^{v_i+1} c_{\mu_i-1}(V_{out}/V_i')).$$

Now, the fibers of $r_1$ are projective spaces $\mathbb{P}^{m_i}$, and $V_i'/V_i$ pulls back to the line bundle $\mathcal{O}(-1)$, so $z$ corresponds to $-H$, and the integral of $z^{m_i}$ is $(-1)^{m_i}$, as desired. □

While the Cautis-Lauda and geometric dualities are different, they induce the same double dual map:

**Proposition 2.2.** *The double dual maps of the Cautis-Lauda and geometric biadjunctions agree. In particular, the $\beta$-geometric duality is cyclic.*

*Proof.* For the double dual of $y$, this is clear. For $\psi$, the consider the diagram [CL15, 2.5]. The 2 cups and 2 caps of this diagram contribute a factor of $-1$ raised to

$$\xi_j(\mu) + \xi_i(\mu - \alpha_j) - \xi_i(\mu) - \xi_j(\mu - \alpha_i) = -v_j - \langle \alpha_i, \alpha_j \rangle - v_i + v_i + \langle \alpha_j, \alpha_i \rangle + v_j = 0.$$

This completes the proof. □

Since this representation sends grading shift to shift in the dg-category of $\mathcal{A}_\mu$-modules, it induces a current algebra action on the center of this category in the dg-sense: its Hochschild cohomolgy $HH(\mathcal{A}_\mu)$.

As discussed before and in [BLPW, §5.4], we have a natural map $HH(\mathcal{A}_\mu) \to Z(\mathrm{Ext}^\bullet(M, M))$ for any sheaf of $\mathcal{A}_\mu$-modules $M$. By [Weba, 3.6], there is a semi-simple core module $C_\mu$ such that $\mathrm{Ext}^\bullet(C_\mu, C_\mu) \cong R^\lambda_\mu$, with this isomorphism inducing an equivalence of dg-categories between the categories $\mathcal{C}^\lambda_\mu$ of core modules and $R^\lambda_\mu$-dg-mod compatible with the categorical actions of $\mathcal{C}^\lambda_\mu$ and $R^\lambda_\mu$-dg-mod. That is:

**Corollary 2.3.** *For any cyclic duality, the map $H^*(\mathfrak{M}^\lambda_\mu) \cong HH(\mathcal{A}_\mu) \to Z(\mathrm{Ext}^\bullet(C_\mu, C_\mu)) \cong Z(R^\lambda_\mu)$ is compatible with the current algebra action induced by that duality on both sides.*

This is half of our main theorem, since it relates the current algebra action of $Z(R^\lambda_\mu)$ to the cohomology of $H^*(\mathfrak{M}^\lambda_\mu)$. However, it is not obvious that the action of $H^*(\mathfrak{M}^\lambda_\mu)$ induced by the $\mathcal{U}$-action agrees with Varagnolo's. We turn to showing this in the next section.

## 3. Hochschild cohomology and Betti cohomology

3.1. **Hochschild vs. Betti.** Now let us discuss the connection of the categorical action $\mathcal{G}_\lambda$ to the usual pushforward and pullback operations on cohomology. As before, let $\mathfrak{P}_i \subset \mathfrak{M}^\lambda_{\mu+\alpha_i} \times \mathfrak{M}^\lambda_\mu$ be the Hecke correspondence for $i$; let $U$ be the open complement of this variety. This subvariety has projection maps

$$\pi_1 \colon \mathfrak{P}_i \to \mathfrak{M}^\lambda_{\mu+\alpha_i} \qquad \pi_2 \colon \mathfrak{P}_i \to \mathfrak{M}^\lambda_\mu.$$

For simplicity, let $d_\mu = \dim_{\mathbb{C}} \mathfrak{M}^\lambda_\mu$. There is the usual pullback map in cohomology $\pi_i^*$, but we'll also wish to consider the pushforwards

$$(\pi_1)_* \colon H^*(\mathfrak{P}_i) \cong H^{BM}_{d_\mu + d_{\mu+\alpha_i} - *}(\mathfrak{P}_i) \to H^{BM}_{d_\mu + d_{\mu+\alpha_i} - *}(\mathfrak{M}^\lambda_{\mu+\alpha_i}) \cong H^{-d_\mu + d_{\mu+\alpha_i} + *}(\mathfrak{M}^\lambda_{\mu+\alpha_i}).$$

$$(\pi_2)_*\colon H^*(\mathfrak{P}_i) \cong H^{BM}_{d_\mu+d_{\mu+\alpha_i}-*}(\mathfrak{P}_i) \to H^{BM}_{d_\mu+d_{\mu+\alpha_i}-*}(\mathfrak{M}^\lambda_\mu) \cong H^{d_\mu-d_{\mu+\alpha_i}+*}(\mathfrak{M}^\lambda_\mu).$$

By definition, Varagnolo's current algebra operators [Var00, (4.1)] depend on a choice of $\beta$, and are given by:

$$x^+_{i,r}(m) = (-1)^{\beta(\mu-\lambda,\alpha_i)}(\pi_1)_*(z^r \cup \pi_2^* m) \qquad x^-_{i,r}(m) = (-1)^{\beta(\alpha_i,\mu-\lambda)}(\pi_2)_*(z^r \cup \pi_1^* m)$$

where as above, $z = c_1(V'_i/V_i)$ is the Chern class of the tautological line bundle over the Hecke correspondence. Note that Varagnolo uses a more restricted set of $\beta$'s, but one can easily check that changing this function by a symmetric bilinear form only changes the action by an automorphism of the Yangian.

**Lemma 3.1.** *We have an isomorphism* $\mathrm{Ext}^\bullet(\mathscr{E}_i,\mathscr{E}_i) \cong H^*(\mathfrak{P}_i;\mathbb{C})$ *and the natural algebra maps*

$$HH(\mathcal{A}_{\mu+\alpha_i}) \to \mathscr{E}xt^\bullet(\mathscr{E}_i,\mathscr{E}_i) \leftarrow HH(\mathcal{A}_\mu)$$

*are intertwined with the pullbacks* $\pi_1^*, \pi_2^*$.

*Proof.* The proof closely follows the computation of the Hochschild cohomology of $\mathcal{A}$ in [BLPW, §5.4]. Since $\mathfrak{P}_i$ is smooth, computing the sheaf Ext $\mathscr{E}xt^\bullet(\mathscr{E}_i,\mathscr{E}_i)$ reduces to a local computation with the vacuum representation of the Weyl algebra. This has vanishing higher self-Exts and 1 dimensional in degree 0, as the usual Koszul resolution shows. Thus, $\mathscr{E}xt^\bullet(\mathscr{E}_i,\mathscr{E}_i) \cong \mathbb{C}_{\mathfrak{P}_i}$, and so $\mathrm{Ext}^\bullet(\mathscr{E}_i,\mathscr{E}_i) \cong H^*(\mathfrak{P}_i;\mathbb{C})$. This local computation also shows the match with pullback in cohomology, since they must be given by the unique map of algebra sheaves between $\pi_1^*\mathbb{C}_{\mathfrak{M}^\lambda_{\mu+\alpha_i}}$ (resp. $\pi_2^*\mathbb{C}_{\mathfrak{M}^\lambda_\mu}$) and $\mathbb{C}_{\mathfrak{P}_i}$. □

If we fix a duality as before, then we also have pushforward maps

$$HH(\mathcal{A}_{\mu+\alpha_i}) \xleftarrow{\tau} \mathrm{Ext}^\bullet(\mathscr{E}_i,\mathscr{E}_i) \xrightarrow{\tau^\star} HH(\mathcal{A}_\mu),$$

as defined in Section 1.2.

**Lemma 3.2.** *Under the geometric duality, the maps $\tau$ (resp. $\tau^\star$) are intertwined with the pushforward maps in cohomology $(\pi_1)_*$ (resp. $(\pi_2)_*$).*

*Proof.* First, note that both maps are induced by sheaf maps from the pushforward $(\pi_1)_*\mathbb{C}_{\mathfrak{P}_i}$ (resp. $(\pi_2)_*\mathbb{C}_{\mathfrak{P}_i}$) of the constant sheaf on $\mathfrak{P}_i$ to the constant sheaf $\mathbb{C}_{\mathfrak{M}^\lambda_{\mu+\alpha_i}}$ (resp. $\mathbb{C}_{\mathfrak{M}^\lambda_\mu}$).

Since we have fixed $i$, we can choose an orientation for which $i$ is a source. In this case, $\mathfrak{M}^\lambda_\mu$ is a open subset of the stack quotient $T^*\hat{X}^\lambda_\mu$, and $\mathfrak{P}_i$ the corresponding open subset of the conormal $N^*\hat{X}^\lambda_{\mu;\alpha_i}$. The functors $\mathscr{E}_i$ and $\mathscr{F}_i$ are induced by pushforward and pullback in the category of D-modules along the maps $\hat{X}^\lambda_\mu \leftarrow \hat{X}^\lambda_{\mu;\alpha_i} \to \hat{X}^\lambda_{\mu+\alpha_i}$. The geometric duality precisely matches under Riemann-Hilbert correspondence with the usual biadjunction for the corresponding functors on constructible sheaves. That is, it corresponds to the natural sheaf maps

$$(\pi_1)_*\mathbb{C}_{\hat{X}^\lambda_{\mu;\alpha_i}} \to \mathbb{C}_{\hat{X}^\lambda_{\mu+\alpha_i}}[d_{\mu+\alpha_i} - d_\mu] \qquad (\pi_2)_*\mathbb{C}_{\hat{X}^\lambda_{\mu;\alpha_i}} \to \mathbb{C}_{\hat{X}^\lambda_\mu}[d_\mu - d_{\mu+\alpha_i}]. \tag{4}$$

On the other hand, Varagnolo's construction proceeds using similar maps

$$(\pi_1)_*\mathbb{C}_{\mathfrak{P}_i} \to \mathbb{C}_{\mathfrak{M}^\lambda_{\mu+\alpha_i}}[d_{\mu+\alpha_i} - d_\mu] \qquad (\pi_2)_*\mathbb{C}_{\mathfrak{P}_i} \to \mathbb{C}_{\mathfrak{M}^\lambda_\mu}[d_\mu - d_{\mu+\alpha_i}],$$

which are in turn the restriction to the stable locus of sheaf maps

$$(5)\qquad (\pi_1)_*\mathbb{C}_{N^*\hat{X}^\lambda_{\mu;\alpha_i}} \to \mathbb{C}_{T^*\hat{X}^\lambda_{\mu+\alpha_i}}[d_{\mu+\alpha_i}-d_\mu] \qquad (\pi_2)_*\mathbb{C}_{N^*\hat{X}^\lambda_{\mu;\alpha_i}} \to \mathbb{C}_{T^*\hat{X}^\lambda_{\mu}}[d_\mu-d_{\mu+\alpha_i}].$$

While these varieties are different, we have vector bundle map

$$q_1\colon T^*\hat{X}^\lambda_{\mu+\alpha_i} \to \hat{X}^\lambda_{\mu+\alpha_i} \qquad q_2\colon T^*\hat{X}^\lambda_{\mu+\alpha_i} \to \hat{X}^\lambda_{\mu+\alpha_i} \qquad q\colon N^*\hat{X}^\lambda_{\mu;\alpha_i} \to \hat{X}^\lambda_{\mu;\alpha_i}$$

such that

$$(q_1)_*\mathbb{C}_{T^*\hat{X}^\lambda_{\mu+\alpha_i}} \cong \mathbb{C}_{\hat{X}^\lambda_{\mu+\alpha_i}} \qquad (q_2)_*\mathbb{C}_{T^*\hat{X}^\lambda_{\mu}} \cong \mathbb{C}_{\hat{X}^\lambda_{\mu}} \qquad q_*\mathbb{C}_{N^*\hat{X}^\lambda_{\mu;\alpha_i}} \cong \mathbb{C}_{\hat{X}^\lambda_{\mu;\alpha_i}}$$

and the pushforward map intertwines the maps of (4) and of (5) by the usual functoriality.

Thus, the map of sheaves induced by $\hat{\mathscr{E}}_i$ and $\hat{\mathscr{F}}_i$ agrees the pushforward and pullback of constructible sheaves on these larger varieties. Since this is a local property, it is unchanged by pulling back to the open subsets $\mathfrak{M}^\lambda_\mu$, $\mathfrak{P}_i$, and $\mathfrak{M}^\lambda_{\mu+\alpha_i}$. □

Thus, immediately from the definition, we have that:

**Corollary 3.3.** *The isomorphism $H^*(\mathfrak{M}^\lambda_\mu) \to HH(\mathcal{A}_\mu)$ intertwines the action of the current algebra for any function $\beta$, and the action induced by $\boldsymbol{\mathcal{U}}$ with the $\beta$-geometric duality.*

This allows to complete the proof of the first part of our main theorem:

**Theorem 3.4.** *The induced map $\phi\colon H^*(\mathfrak{M}^\lambda_\mu) \to Z(R^\lambda_\mu)$ is injective.*

*Proof.* Choose $\beta'$ so that the $\beta$-geometric and $\beta'$-Cautis-Lauda dualities agree. Combining Corollaries 2.3 and 3.3 show that $\phi$ intertwines the $\beta$-action on $H^*(\mathfrak{M}^\lambda_\mu)$ with the $\beta'$-Cautis-Lauda action on $Z(R^\lambda_\mu)$.

As argued in [KN12, 4.4], following [Nak01, 13.3.1], the homology $\bigoplus_\mu H^{BM}_*(\mathfrak{M}^\lambda_\mu)$ is cyclic as a $\mathfrak{g}[t]$-module and generated by fundamental class of $\mathfrak{M}^\lambda_\lambda$. Dually, this means that $H^*(\mathfrak{M}^\lambda_\mu)$ is cocyclic, and cogenerated by the identity in $H^*(\mathfrak{M}^\lambda_\lambda) \cong \mathbb{C}$. Thus, any map which is compatible with $\mathfrak{g}[t]$ either kills this vector or is injective. The former is clearly false, since $\phi$ in weight $\lambda$ is an algebra map between copies of $\mathbb{C}$. Thus $\phi$ is injective. □

3.2. **Surjectivity.** Let us consider how the maps of Theorem 3.4 interacts with Kirwan surjectivity. For each $\mathbf{v}$, we have a Kirwan map $\kappa_g\colon H^*_{G_\mathbf{v}}(*) \to H^*(\mathfrak{M}^\lambda_\mu)$; the image of this map is the same as the subring generated by the Chern classes of the tautological bundles $V_i$. Note that we have a natural map $\rho\colon H^*_{G_\mathbf{v}}(*) \to Z(R_{\lambda-\mu})$ to the center of the KLR algebra; this induced by the identification of VV: $R_{\lambda-\mu} \to {}^\delta\mathbf{Z}_\mathbf{v}$ with a convolution algebra ${}^\delta\mathbf{Z}_\mathbf{v}$ in [VV11, 3.6]. Explicitly, this sends the $k$th Chern class $c_{i,k}$ of the associated bundle $V_i \times^{G_\mathbf{v}} EG_\mathbf{v}$ to the degree $k$ elementary symmetric polynomial on the strands with label $i$. Note that by [KL09, 2.9], this means that $\rho$ is an isomorphism.

**Proposition 3.5.** *The following diagram commutes:*

$$\begin{tikzcd} H^*_{G_{\mathbf{v}}}(*) \arrow[r, "\rho"] \arrow[d, "\kappa_g"'] & Z(R_{\lambda-\mu}) \arrow[d, "\kappa_a"] \\ H^*(\mathfrak{M}^\lambda_\mu) \arrow[r, "\phi"'] & Z(R^\lambda_\mu) \end{tikzcd}$$

*Proof.* This commutation is essentially automatic from the definition. We can fill in a middle term in this commutative diagram:

$$\begin{tikzcd} H^*_{G_{\mathbf{v}}}(*) \arrow[r, "\rho'"] \arrow[d, "\kappa_g"'] & Z({}^\delta\mathbf{Z}_{\mathbf{v}}) \arrow[d] & Z(R_{\lambda-\mu}) \arrow[l, "Z(\mathrm{VV})"'] \arrow[d, "\kappa_a"] \\ H^*(\mathfrak{M}^\lambda_\mu) \arrow[r, "\phi'"'] & Z(\mathrm{End}(C_\mu)) & Z(R^\lambda_\mu) \arrow[l] \end{tikzcd}$$

The map $R^\lambda_\mu \to \mathrm{End}(C_\mu)$ is precisely defined by taking the image of VV under a reduction functor, so the right hand square is commutative by definition. The left hand square can thought of as follows. We can identify ${}^\delta\mathbf{Z}_{\mathbf{v}}$ with the endomorphisms of the sheaf $L_{\mathbf{v}} := \bigoplus_{|\mathbf{i}|=\mathbf{v}} L \star \tilde{\mathscr{E}}_{i_1} \star \cdots \star \tilde{\mathscr{E}}_{i_n}$ on $T^*X^\lambda_\mu$ where $L$ is the pushforward of functions on the subspace $\oplus_i \mathrm{Hom}(V_i, W_i) \subset E^\lambda_\mu$; taking solution sheaf, we obtain the pullback of Varagnolo and Vasserot's sheaf ${}^\delta\mathcal{L}_{\mathbf{i}}$ via the projection $E^\lambda_\mu \to E^0_{\mu-\lambda}$, so the endomorphism are the same.

On the other hand, $C_\mu$ is the image of $L_{\mathbf{v}}$ under the Kirwan functor, with this functor inducing the middle vertical map. In this context, we can think of the Kirwan functor as pullback from $T^*X^\lambda_\mu$ to $\mathfrak{M}^\lambda_\mu$ of a microlocal D-module. Thus, we can think of the map $\rho'$ as the action induced by the identification $H^*(T^*X^\lambda_\mu) \cong H^*_{G_{\mathbf{v}}}(*)$, induced the natural map of sheaves $\mathbb{C}_{T^*X^\lambda_\mu} \to \mathcal{E}nd(\boldsymbol{\mu}L_{\mathbf{v}})$.

Since both the left and middle vertical maps are induced by pullback $\mathfrak{M}^\lambda_\mu \subset T^*X^\lambda_\mu$, and the maps $\rho', \phi'$ by the map from Betti cohomology to Hochschild cohomology, the square commutes by naturality of this map. □

Thus, under $\phi$, the images of $\kappa_g$ and $\kappa_a$ coincide.

**Corollary 3.6.** *If algebraic Kirwan surjectivity (i.e. the surjectivity of $\kappa_a$) holds, then the map $\phi$ is an isomorphism and the map $\kappa_g$ is surjective. That is, the conventional Kirwan surjectivity theorem holds for $\mathfrak{M}^\lambda_\mu$.*

Combining with Proposition 1.6 completes the proof of Theorem A. It feels slightly embarrassing to the author to write this as a corollary, since it very easily seen directly from [KN12] in the finite case, but for the sake of posterity, let us note:

**Corollary 3.7.** *Kirwan surjectivity holds for quiver varieties of finite ADE.*

Unfortunately, it's not clear that this algebraic analog of Kirwan surjectivity is any easier to prove in more general situations.

3.3. **Generalizations.** We can give a small generalization of this theorem: there is a generalization of $C_\mu$ to a category $\mathcal{O}$ over the quiver variety $\mathfrak{M}^\lambda_\mu$ based on the choice of a Hamiltonian $\mathbb{C}^*$-action. In [BLPW, 5.23], we conjecture that for any semi-simple generator $O$ of this category, the map $H^*(\mathfrak{M}^\lambda_\mu) \to Z(\operatorname{Ext}^\bullet(O, O))$ is an isomorphism. We can now show an important special case of this conjecture:

**Proposition 3.8.** *If $\Gamma$ is a finite ADE quiver and $O$ a semi-simple generator of category $\mathcal{O}$ for any $\mathbb{C}^*$-action, then the map $H^*(\mathfrak{M}^\lambda_\mu) \to Z(\operatorname{Ext}^\bullet(O, O))$ is an isomorphism*

*Proof.* By [Webc, Th. A], we have that $\operatorname{Ext}^\bullet(O, O)$ is Morita equivalent to a tensor product algebra $T^{\underline{\lambda}}$ in finite type. In both cases, the modules over these algebras are covers of $R^\lambda_\mu$-mod by [Webb, Th. C]; that is, the projective modules over $\operatorname{Ext}^\bullet(O, O)$ embed as a subcategory of $R^\lambda_\mu$-mod. This implies that these algebras have the same center, completing the proof. $\square$

Theorem 3.4 can also be upgraded to a $K = G_{\mathbf{w}}$-equivariant version in finite type, and $K = G_{\mathbf{w}} \times \mathbb{C}^*$-equivariant with the $\mathbb{C}^*$ acting by weight 1 on the edges of the cycle in affine type A. Let $\check{R}^\lambda_\mu$ be the deformed KLR algebra defined in [Webb, §3.5]; if $\Gamma$ has affine type A, then let $\check{\mathbb{R}}^\lambda_\mu$ be the deformed KLR algebra over $\mathbb{C}[h]$ attached to the polynomials

$$Q^h_{ij}(u, v) = (u - v + h)^{\#\{j \to i\}} (v - u + h)^{\#\{i \to j\}}.$$

**Proposition 3.9.** *There is an injective algebra map*

$$H^*_{G_{\mathbf{w}}}(\mathfrak{M}^\lambda_\mu) \to Z(\check{R}^\lambda_\mu) \cong \operatorname{Ext}^\bullet_{G_{\mathbf{w}}}(O, O)$$

*which is compatible with the current algebra actions. This map is an isomorphism for finite type ADE. In affine type A, we can deform further to an isomorphism*

$$H^*_{G_{\mathbf{w}} \times \mathbb{C}^*}(\mathfrak{M}^\lambda_\mu) \to Z(\check{\mathbb{R}}^\lambda_\mu) \cong \operatorname{Ext}^\bullet_{G_{\mathbf{w}} \times \mathbb{C}^*}(O, O).$$

*Proof.* First, we need to that these maps exist. We have a natural map from $H^*_K(\mathfrak{M}^\lambda_\mu)$ to the Ext algebra of $C_\mu$ in the $K$-equivariant derived category. This equivariant Ext-algebra must be a base extension of $\check{R}^\lambda_\mu$ by [Webb, 3.27]; this base extension can calculated using the minimal polynomial of $y$ acting on $C_{\lambda - \alpha_i}$; since the quiver variety in thise case in $T^*\mathbb{P}(W_i)$, we have that the only core module can be identified with the zero-section, and its equivariant Ext-algebra is just

$$H^*_{G_{\mathbf{w}}}(\mathbb{P}(W_i)) \cong \mathbb{C}[y]/(y^{w_i} - c_{i,1} y^{w_i - 1} + \cdots + (-1)^{w_i} c_{i,w_i})$$

where $c_{i,j}$ is the $j$th Chern class of the tautological bundle $W_i$ on $*/G_{\mathbf{w}}$. Thus, we obtain the universal quantization itself, with the deformation parameters precisely matching the Chern classes on $*/G_{\mathbf{w}}$.

In order to extend to the affine deformation, we simply note that the KLR algebra is deformed to that for $Q^h_{ij}(u, v)$ by the addition of $\mathbb{C}^*$-action; otherwise, the proof is the same. $\square$

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